Solving Systems of Equations

1. Solve this system, and write its solution set *both* in parametric form *and* in parametric vector form.

$$\begin{cases} x_1 + \qquad x_3 + 2x_4 = 3 \\ \qquad x_2 + x_3 + 3x_4 = 2 \\ 2x_1 + -x_2 + x_3 + x_4 = 4 \end{cases}$$

reduce

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & -1 & -1 & -3 & -2 \end{array}\right] r_3 - 2r_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] r_3 + r_2$$

$$\Longleftrightarrow \begin{cases} x_1 \qquad + x_3 + 2x_4 = 3 \\ x_2 + x_3 + 3x_4 = 2 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_1 = 3 - x_3 - 2x_4 \\ x_2 = 2 - x_3 - 3x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{cases}$$

Solutions

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 - x_3 - 2x_4 \\ 2 - x_3 - 3x_4 \\ 0 + x_3 + 0 \\ 0 + 0 + x_4 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

for all x_3, x_4 in $\mathbb{R}$

Practice Quiz 1 Math 202

Name: Solutions. Section: ______

You have [redacted] minutes to complete the quiz. Please **show all work**, and then **circle your answer.**

1. Fill in the blanks, to complete the statement of Theorem 2:

   Theorem 2: The reduced echelon form of a linear system has three possible cases

   (a) The system has zero solutions if it contains a row $[0 \dots 0 | \blacksquare]$

   (b) The system has exactly one solutions if it is consistent and has a pivot in every coeff. column.

   (c) The system has ∞-many solutions if it is consistent and if some coeff column does NOT contain a pivot

2. For each of the cases above, write down an augmented matrix with the corresponding number of solutions.

   (a) zero

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

requires $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 4$ but $0 \neq 4$

   (b) unique

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

   (c) ∞-many

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(free variable in column 2)

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4. (No Computation) For each of the following matrices, determine if its augmented matrix is in echelon form, reduced echelon form, or neither. If it is in echelon form, indicate which *columns* contain pivots.

(a)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Not in either - pivots are not stepping in order

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reduced echelon form
pivot in <u>every</u> column

(c)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

echelon form but <u>NOT</u> reduced echelon form
pivots in columns 1, 2, & 3.

(d)

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reduced echelon form.
pivots in columns 1, 2, 4.

Name: Solutions Section: ____

1. (a) Write the following augmented matrix as a vector equation, a matrix equation, and a system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 1 \end{array}\right]$$

$$x_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 + 2x_2 - x_3 = -3 \\ 2x_1 + 3x_2 + x_3 = 1 \end{cases}$$

3 cases $\begin{cases} \text{2pt for all 3} \\ \text{1pt for 2} \\ \text{0.5 pt for 1.} \end{cases}$

(b) Is the system of equations consistent?

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -1 & 3 & 7 \end{array}\right] r_2 - 2r_1$$

← echelon form doesn't contain $[0 \ldots 0 \mid \blacksquare]$ ⇒ consistent

1pt

1pt

(c) Solve the above system. (Find the solution set).

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 11 \\ 0 & -1 & 3 & 7 \end{array}\right]^{r_1 + 2r_2} \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 11 \\ 0 & 1 & -3 & -7 \end{array}\right]$$

2pt for final matrix

$$\Longleftrightarrow \begin{cases} x_1 \quad + 5x_3 = 11 \\ x_2 \quad - 3x_3 = -7 \\ x_3 \text{ free} \end{cases} \Longleftrightarrow \begin{cases} x_1 = 11 - 5x_3 \\ x_2 = -7 + 3x_3 \\ x_3 \text{ free} \end{cases}$$

2pt for parametric form

(d) If the system is consistent, write down a particular solution.

Verify that this is a solution *two different ways* by plugging it into both *i.* the vector equation, and *ii.* the system of equation.

Pick $x_3 = 0$

$$\Rightarrow \vec{x} = \begin{bmatrix} 11 \\ -7 \\ 0 \end{bmatrix}$$

$$11 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-7)\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 0 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 22 \end{bmatrix} + \begin{bmatrix} -14 \\ -21 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \checkmark$$

$$\begin{cases} 11 + 2\cdot(-7) - 0 = 11 - 14 = -3 \checkmark \\ 2\cdot 11 + 3\cdot(-7) + 0 = 22 - 21 = 1 \checkmark \end{cases}$$

← 1pt each

Name: ______________________ Section: ______________________

2. (a) Write the following augmented matrix as a vector equation, a matrix equation, and a system of linear equations.

$$\left[\begin{array}{cc|c} 0 & 4 & 6 \\ 5 & 2 & 10 \\ 3 & 3 & 6 \end{array}\right]$$

$$x_1 \cdot \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 \\ 5 & 2 \\ 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 6 \end{bmatrix}$$

$$\begin{cases} 4x_2 = 6 \\ 5x_1 + 2x_2 = 10 \\ 3x_1 + 3x_2 = 6 \end{cases}$$

(b) Is the system of equations consistent?

$$\sim \left[\begin{array}{cc|c} 3 & 3 & 6 \\ 5 & 2 & 10 \\ 0 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 5 & 2 & 10 \\ 0 & 2 & 3 \end{array}\right] \begin{array}{l} \tfrac{1}{3}r_1 \\ \\ \tfrac{1}{2}r_3 \end{array} \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 2 & 3 \end{array}\right] r_2 - 5r_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 3 \end{array}\right] -\tfrac{1}{3}r_2 \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{array}\right] r_3 - 2r_1$$

INCONSISTENT!

requires $0 = 3$

(c) Solve the above system. (Find the solution set).

No solutions

(empty set)

(d) If the system is consistent, write down a particular solution.

Verify that this is a solution *three different ways* by plugging it into *i.* the vector equation, *ii.* matrix equation, and *iii.* system of equation.

N/A

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3. (a) Write the following augmented matrix as a vector equation, a matrix equation, and a system of linear equations.

$$\left[\begin{array}{ccc|c} 3 & -3 & 3 & 3 \\ -1 & 2 & 1 & 3 \\ 2 & 8 & 2 & 2 \end{array}\right]$$

$$x_1 \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -3 \\ 2 \\ 8 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{cases} 3x_1 - 3x_2 + 3x_3 = 3 \\ -x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + 8x_2 + 2x_3 = 2 \end{cases}$$

$$\begin{bmatrix} 3 & -3 & 3 \\ -1 & 2 & 1 \\ 2 & 8 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$$

(b) Is the system of equations consistent?

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & 2 & 1 & 3 \\ 1 & 4 & 1 & 1 \end{array}\right] \begin{array}{l} \tfrac{1}{3}r_1 \\ \\ \tfrac{1}{2}r_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 5 & 0 & 0 \end{array}\right] \begin{array}{l} \\ r_2 + r_1 \\ r_3 - r_1 \end{array}$$

Step form

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

consistent because omits [0 0 0 |

(c) Solve the above system. (Find the solution set).

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} r_1 + r_2 \\ \\ \tfrac{1}{2}r_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} r_1 - r_3 \\ \\ \\ \end{array}$$

$$\begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 2 \end{cases}$$

(d) If the system is consistent, write down a particular solution.

Verify that this is a solution *three different ways* by plugging it into *i.* the vector equation, *ii.* matrix equation, and *iii.* system of equation.

~~vector~~ equation
Vector

$$-1 \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + 0 \cdot \begin{bmatrix} -3 \\ 2 \\ 8 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \checkmark$$

system of eqns

$$\begin{cases} 3 \cdot (-1) + -3 \cdot 0 + 3 \cdot 2 = -3 + 6 = 3 \checkmark \\ -1(-1) + 2 \cdot 0 + 1 \cdot 2 = 1 + 2 = 3 \checkmark \\ 2(-1) + 8 \cdot 0 + 2 \cdot 2 = -2 + 4 = 2 \checkmark \end{cases}$$

Exam I Review Math 202

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Vectors and Vector Equations

1. (No Computation) Write down the formal definition of Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{array}{l} \text{the set of all} \\ \vec{b} = c_1 \cdot \vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \ : \text{ for } c_1, c_2, c_3 \text{ in } \mathbb{R}. \end{array} \right\}$$

2. (No Computation) What is the graphical meaning of Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$?

it is the set of all vectors that can be hit by scaling & adding $\vec{v}_1$, $\vec{v}_2$, & $\vec{v}_3$

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3. Describe the Span of the following sets of vectors:

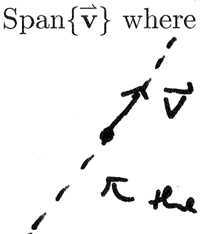
(a) Span$\{\vec{\mathbf{v}}\}$ where $\vec{\mathbf{v}} = \vec{\mathbf{0}}$

$\{t \cdot \vec{0} : t \text{ is in } \mathbb{R}\} = \{\vec{0}\}$

• a single point

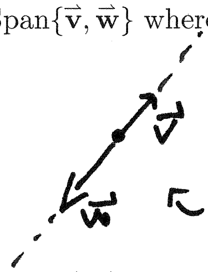
set containing ONLY the zero vector.

(b) Span$\{\vec{\mathbf{v}}\}$ where $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$



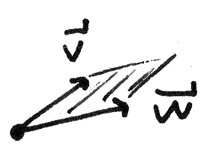
the line through $\vec{v}$ and $\vec{0}$

(c) Span$\{\vec{\mathbf{v}}, \vec{\mathbf{w}}\}$ where $\vec{\mathbf{v}}, \vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{w}} = k\vec{\mathbf{v}}$ for some $k \in \mathbb{R}$.



still just a line through $\vec{v}$ and $\vec{0}$.

(d) Span$\{\vec{\mathbf{v}}, \vec{\mathbf{w}}\}$ where $\vec{\mathbf{v}}, \vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{w}} \neq k\vec{\mathbf{v}}$ for every $k \in \mathbb{R}$.



the plane containing $\vec{0}$, $\vec{v}$, and $\vec{w}$.

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4. Is the vector $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ in the span of the vectors $\vec{\mathbf{a}}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\vec{\mathbf{a}}_2 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$, $\vec{\mathbf{a}}_3 = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$?

i.e. is $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$ consistent?

reduce $\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -3 & -3 & 3 \\ -2 & 2 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 1 & -1 & -1 & 1 \\ -2 & 2 & 1 & 6 \end{array}\right]$ $\tfrac{1}{3}r_2$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -3 & 10 \end{array}\right]$ $\begin{array}{l} \\ r_2 - r_1 \\ r_3 + 2r_1 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 7 \end{array}\right]$ $r_3 + 3r_2$

No solution, by theorem 2

So vector is NOT in span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

5. Find h so that the vector the vector $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ h \\ 3 \end{bmatrix}$ in the span of the vectors

$\vec{\mathbf{a}}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{\mathbf{a}}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{\mathbf{a}}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

want $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & -1 & 0 & h \\ -1 & 1 & 0 & 3 \end{array}\right]$ to be consistent

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -2 & -2 & h-2 \\ 0 & 2 & 2 & 5 \end{array}\right]$ $\begin{array}{l} \\ r_2 - r_1 \\ r_3 + r_1 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -2 & -2 & h-2 \\ 0 & 0 & 0 & h+3 \end{array}\right]$ $r_3 + r_2$

this is consistent $\Leftrightarrow h + 3 = 0$

$\Leftrightarrow h = -3$

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6. Write the Solution Set of the equation

$$x_1 \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -1 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ \cancel{-4} \\ -4 \end{bmatrix} 2$$

in parametric vector form.

reduce

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 4 \\ -2 & 1 & -1 & \cancel{-4}\,2 \\ -4 & -2 & -6 & -4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 2 & 4 & 4 \\ 0 & 3 & 3 & 6 \\ 0 & 2 & 2 & 4 \end{array}\right] \begin{array}{l} \\ r_2 + r_1 \\ r_3 + 2r_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right] \begin{array}{l} 1/2\, r_1 \\ 1/3\, r_2 \\ 1/2\, r_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1 - r_2 \\ \\ r_3 - r_2 \end{array}$$

$\Leftrightarrow$

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 2 \\ x_3 \text{ free} \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} x_1 = -x_3 \\ x_2 = 2 - x_3 \\ x_3 \text{ free} \end{cases}$$

Solutions are

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 - x_3 \\ 2 - x_3 \\ 0 + x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

for all x_3 in $\mathbb{R}$

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7. (No Computation) Find 4 vectors that do span $\mathbb{R}^3$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$$

8. (No Computation) Find 4 vectors that do NOT span $\mathbb{R}^3$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}$$

9. (No Computation) Can you find 3 vectors that DO span $\mathbb{R}^4$? Why or why not?

<u>No.</u> the columns of

$[\vec{a}_1, \vec{a}_2, \vec{a}_3]$ span $\mathbb{R}^4$

$\Leftrightarrow$

there is a pivot in each row by theorem 4.

<u>But</u> you cannot have 4 pivots in 3 columns.

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Matrices and Matrix Equations

1. If $A = [\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n]$ and $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, write down the formal definition of the matrix product

$$A\vec{\mathbf{x}} = [\vec{\mathbf{a}}_1, \ldots, \vec{\mathbf{a}}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot \vec{a}_1 + \cdots + x_n \cdot \vec{a}_n$$

2. Compute the following matrix products. If the product is undefined, explain why.

(a) $\begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ undefined. need # columns of A = # rows of $\vec{x}$

(b) $\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}$

3. (No Computation) Suppose that the $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has the form $\vec{\mathbf{x}} = t\vec{\mathbf{v}} + s\vec{\mathbf{w}}$. Suppose also that $A\vec{\mathbf{p}} = \vec{\mathbf{b}}$. Describe the full solution set for $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

$A\vec{p} = \vec{b} \Rightarrow \vec{p}$ is a solution to $A\vec{x} = \vec{b}$

$\Rightarrow$ solution set to $A\vec{x} = \vec{b}$

$= \{ \vec{w} = \vec{p} + \underbrace{t \cdot \vec{v} + s \cdot \vec{w}}_{\text{sln to } A\vec{x}=0} : t, s \text{ in } \mathbb{R} \}$

What is the geometric relationship between the solution set for $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ and the solution set for $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$?

The solution set of $A\vec{x} = \vec{b}$

=

the solution set of $A\vec{x} = \vec{0}$ translated by some $\vec{p}$ s.t. $A\vec{p} = \vec{b}$.

Practice Quiz 1 Math 202

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3. Write down the formal definition of Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

the set of $\vec{b}$ s.t. $\vec{b} = c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + c_3 \cdot \vec{v}_3$ for some c_1, c_2, c_3 in $\mathbb{R}$

4. What is the graphical meaning of Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$?

it is the set of vectors that can be gotten by scaling & adding $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

5. Fill in the blanks to state Theorem 4 in terms of pivots.

Theorem 4: The columns of an $m \times n$ matrix A span $\mathbb{R}^m$

if and only if there is a pivot in every Row of A

6. Write down a 3×3 matrix A whose columns span $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Write down a 3×3 matrix A whose columns do not span $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

8. Can you write down a 2×3 matrix A whose columns span $\mathbb{R}^2$? Justify your answer.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes.
the matrix to the left has a pivot in every Row
$\Rightarrow$ columns span $\mathbb{R}^2$

9. Can you write down a 3×2 matrix A whose columns span $\mathbb{R}^3$? Justify your answer.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

No.
you cannot have 3 pivot rows if you only have two columns.
$\Rightarrow$ no such matrix exist by Theorem 4.

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10. Write down the formal definition of Linear **Dependence** of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly Dependent

If there are c_1, c_2, c_3 in $\mathbb{R}$ NOT all 0
so that

$$c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + c_3 \cdot \vec{v}_3 = \vec{0}$$

11. Give an example of a non-trivial dependence relation between $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$.
Use this dependence relation to explain the graphical meaning of "Linear **Dependence**".

$$2\vec{v}_1 + 6\vec{v}_2 - \vec{v}_3 = \vec{0}$$

solving for $\vec{v}_3 = 2\vec{v}_1 + 6\vec{v}_2$
shows that $\vec{v}_3$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$.

12. What is the graphical meaning of the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ being Linearly **Independent**?

If there is NO nontrivial dependence relation
Then NO $\vec{v}_i$ is in the span of the other remaining vectors.

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4. Determine if the following set of vectors is linearly dependent or independent.

$\vec{\mathbf{a}}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $\vec{\mathbf{a}}_2 = \begin{bmatrix}2\\2\\1\end{bmatrix}$ $\vec{\mathbf{a}}_3 = \begin{bmatrix}-2\\-2\\2\end{bmatrix}$

i.e. is there a nontrivial solution

to $x_1 \cdot \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{0}$?

reduce $\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 1 & 2 & -2 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right] \begin{array}{l} \\ r_2 - r_1 \\ r_3 - r_1 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\leftarrow$ by theorem 2,
3rd column w/o pivot
$\Rightarrow$
∞-many slns
$\Rightarrow$ <u>has</u> nontrivial solution
$\Rightarrow$ $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is
linearly <u>dependent</u>.

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5. Find h so that the vectors $\vec{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$, and $\vec{a}_3 = \begin{bmatrix} 2 \\ h \\ 1 \end{bmatrix}$ are linearly independent [redacted].

want ONLY TRIVIAL solution to

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{0}$$

reduce
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 3 & 4 & h & 0 \\ 2 & 5 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & h-6 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \begin{array}{l} \\ r_2 - 3r_1 \\ r_3 - 2r_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & h-6 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} \\ \\ \tfrac{1}{3} r_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & h-6 & 0 \\ 0 & 0 & \underline{-1-(h-6)} & 0 \end{array}\right] \begin{array}{l} \\ \\ r_3 - r_2 \end{array}$$

NOTICE: has ONLY trivial solution $\Leftrightarrow$ has pivot in column 3

$$\Leftrightarrow -1-(h-6) \neq 0$$
$$-1 - h + 6 \neq 0$$
$$h \neq 5$$

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the set is linearly independent when $h \neq 5$.

Practice Quiz 1 Math 202

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13. Suppose that $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + x_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$ has a unique solution.
Is $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ Linearly **Independent**?

yes. $x_1 = x_2 = x_3 = 0$ is <u>always</u> a solution

<u>So</u> the only dependence relation is the trivial one.

<u>So</u> the vectors are linearly independent

14. Suppose that $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + x_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$ has infinitely many solutions.
Is $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ Linearly **Independent**?

No. ∞-many solutions

$\Rightarrow$ has some nontrivial solution

$\Rightarrow$ there is a nontrivial dependence relation

15. Does $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + x_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$ always have a solution? Why or why not?

yes. $0 \cdot \vec{V}_1 + 0 \cdot \vec{V}_2 + 0 \cdot \vec{V}_3$

$= \vec{0} + \vec{0} + \vec{0}$

$= \vec{0}$

So $x_1 = x_2 = x_3 = 0$ is always a solution.